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# Robust power series algorithm for epistemic uncertainty propagation in Markov chain models

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**ABSTRACT**
In this article, we develop a new methodology for integrating epistemic uncertainties into the computation of performance measures of Markov chain models. We developed a power series algorithm that allows for combining perturbation analysis and uncertainty analysis in a joint framework. We characterize statistically several performance measures, given that distribution of the model parameter expressing the uncertainty about the exact parameter value is known. The technical part of the article provides convergence result, bounds for the remainder term of the power series, and bounds for the validity region of the approximation. In the algorithmic part of the article, an efficient implementation of the power series algorithm for propagating epistemic uncertainty in queueing models with breakdowns and repairs is discussed. Several numerical examples are presented to illustrate the performance of the proposed algorithm and are compared with the corresponding Monte Carlo simulations ones.

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Algorithm; epistemic uncertainty; fundamental matrix; Markov chain; Monte Carlo simulation; power series expansions; queues with breakdowns and repairs

## 1. Introduction

Uncertainty quantification allows us to quantify output variability in the presence of uncertainty. During the past decades, the rapid developments in numerical and computational methods have enabled system characteristics to be quantified under uncertain conditions. Specifically, there is an increasing literature addressing the problem of propagating parameter epistemic uncertainties performance metrics in stochastic models. Some of the newly developed uncertainty propagation methods include Monte Carlo simulation[4,25], random set theory[23], fuzzy set theory[9,21,34], evidence theory method[3,5,33], probabilistic bounding analysis[29], calculation with intervals[15], Bayesian statistical frameworks[26], analytical uncertainty propagation methods[27], perturbation analysis method[6,7], and Taylor series expansion[2,8,24,32]. Among

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these methods, the power series algorithm has attracted increasing attention and is being applied to a wide class of stochastic models. Power series algorithm-based techniques for computational purposes have been used in the past in perturbation analysis of Markov chains; see, for example[10,11,28], and references therein.

Power series expansion for Markov chains is used so as to quantify the uncertainty of a model output as a function of its input parameters. More specifically, most analyses of Markov chains steady-state behavior are focused on the stationary distribution, which plays a key role in determination of other performance measures. In such models, the stationary distribution $\pi$ is obtained as function of some model parameter $\theta$, that is, one obtains $\pi_\theta$. The issue that is addressed in this article is the investigation of uncertainty resulting from the lack of knowledge about the value of the model parameter $\theta$ and the use of a specific computational model for propagating this uncertainty to the model output $\pi_\theta$. To achieve this, we assume that the exact value of the parameter $\theta$ is not precisely known; in practice though, data are available for calibrating the probability distributions that model gaps of knowledge and scarcity of data on $\theta$. We will therefore suppose $\theta$ to be a random variable having the distribution of which is derived from the sample statistic. Conceptually, this gives rise to the following question: *what is the effect of the imperfect knowledge of the parameter $\theta$ on the stationary distribution $\pi_\theta$?* For that, we use robust perturbation analysis in conjunction with the power series expansion methodology and obtain the stationary distribution $\pi_\theta$ under polynomial form in $\theta$ as a computational model.

Many Markov chain models encountered in applied sciences involve input parameters which are not perfectly known. Robust perturbation analysis for Markov chains studies the dependence of the performance measures with respect to the uncertain parameter. This article is concerned with robust perturbation analysis for the discrete-time Markov chains with finite discrete space. More specifically, we develop a new approach to incorporate parameter epistemic uncertainties in computing stationary performance measures of Markov chain models. In probabilistic uncertainty quantification approaches, one represents uncertain model parameters as random variables. Let $(\mathbf{\Omega}, \mathcal{F}, \mathbf{P})$ be the probability space describing the underlying randomness of the uncertain parameter, where $\mathbf{\Omega}$ denotes the event space equipped with $\sigma$-algebra $\mathcal{F}$ and probability measure $\mathbf{P}$. Thereby, we assume that the input parameter $\theta$ is given as follows:

$$\theta(\omega) = \bar{\theta} + \sigma \ \varepsilon(\omega), \tag{1}$$

where $\omega \in \mathbf{\Omega}, \bar{\theta}$ represents the estimated mean value provided by the statistic of $\theta(\omega)$, $\sigma$ is the standard deviation of the same parameter, and $\varepsilon(\omega)$ is a random variable modeling the epistemic distribution. In other words, parameter epistemic uncertainty is represented by a probability distribution. Therefore, the parameter $\theta$ is varied at random, but restricted by the given uncertainty distribution. Common distributions include, but are not limited to, the normal distribution, lognormal distribution, rectangular distribution, and triangular distribution. Once the parameter uncertainty values have been determined, the values can be propagated using different frameworks that have been proposed in the literature to quantify the latter lack of knowledge. Hence, it is important to develop numerical methods for computing performance metrics by incorporating epistemic uncertainties inflicted on parameter values.

In this article, a new methodology based on the power series algorithm is developed to propagate parameter epistemic uncertainties in Markov chain models. This approach is suited for cases in which the stationary distribution and fundamental matrix of the associated Markov chain can be numerically solved, so then we can insert randomness after the numerical evaluation. We illustrate the potential of the proposed approach on the computationally efficient determination of the performance measures of queueing models with breakdowns and repairs, by integrating parameter uncertainty, of the M/G/1/N and the M/M/c/N queues with breakdowns and repairs. The first queue is described by a Markov chain, embedded at departure and repair moments, and the latter queue is governed by a sample chain, embedded at appropriate Poisson times; details will be provided later in the text. The main purpose of this article is to provide a detailed discussion on accurately computing the uncertain performance measures of the Markov chain describing the state of the considered models. These performances will be characterized in a complete and univocal way by estimating their probability density functions. Furthermore, we will show that it is also possible to define some quantities of interest that allow us to specify certain characteristics of these uncertain performance measures. These quantities will not always be sufficient to define them in a univocal way; however, they are very useful when it is necessary to give a synthetic overview on these performances. More specifically, we are interested on computing the mean and the variance of the underlying performance. Furthermore, we investigate the convergence of the power series expansions, and establish an expression for the remainder term of the power series expansions, and discuss the validity region of the approximation

The article is organized as follows. In Section 2, the computation schema of the power series algorithm is stated in a detailed way. This is expressed in terms of the fundamental matrix; a precise definition of the fundamental matrix will be given later. The M/G/1/N and the M/M/c/N queues with breakdowns and repairs are analyzed in Section 3. Eventually, we will point out directions of further research.

## 2. The robust power series expansion

In this section, we propose an efficient computational algorithm for quantifying performance measures of stochastic systems, which can be described by finite-state Markov chains. This algorithm allows us to express the performance measures in closed form as function of system parameters. Specifically, we develop the stationary distribution of the studied model as polynomial in some parameter through a power series expansion. The elements of the power series expansion are derived in terms of the fundamental matrix associated with the underlying Markov chain, which will be defined later in the text.

### 2.1. Approach development

Consider a discrete-time Markov chain $X = \{X_n : n \geq 0\}$ on discrete state space $S = \{0, 1, ..., N-1\}$, which is governed by a transition probability matrix $P$. Throughout this article, we will assume that the Markov chain $X$ is ergodic with unique stationary distribution $\pi = (\pi(0), \pi(1), ..., \pi(N-1))$. This distribution can be found by solving the equation $\pi P = \pi$ and $\pi e = 1$, where $e = (1, 1, ..., 1)^t$ is the unit vector, and the superscript $t$ denotes the transpose. Indeed, $P$ must be a stochastic matrix; that is, $p_{ij} \geq 0$ and $\sum_{j \geq 0} p_{ij} = 1$, for all $i \geq 0$. Denote the ergodic matrix of $P$ by $\Pi$, which has all rows identical and equal to $\pi$. It is easy to see from the uniqueness of the stationary distribution that the matrix $(I-P+\Pi)$ is invertible, where $I$ denotes the identity matrix. Hence, the matrix $Z = (I-P+\Pi)^{-1}$ exists and satisfies:

$$
\begin{aligned}
Z\ \Pi &= \Pi\ Z = \Pi, \\
P\ Z &= Z\ P = Z + \Pi - I.
\end{aligned}
\tag{2}
$$

The fundamental matrix $Z$ is conveniently represented as:

$$
\begin{aligned}
Z &= \sum_{n=0}^{\infty} (P-\Pi)^n, \\
&= I + \sum_{n=1}^{\infty} (P^n - \Pi).
\end{aligned}
\tag{3}
$$

The fundamental matrix plays an important role in perturbation analysis of Markov chains, where it is considered as a tool which measures the speed of convergence with which a transition probability function approaches its limiting value. Especially, for ergodic Markov chains, the fundamental matrix is interesting because it gives access to some quantities such as the mean time to return to state $j$ from state $i$. The research on fundamental matrix was mainly initiated by Kemeny and Snell in Ref.[17]

for discrete-time finite-state Markov chains and afterwards extended to a continuous-time setting[16] and to denumerable state spaces[18]. For more information on fundamental matrix, see, for example[1,12–14,20,22,30].

The basic concept of power series expansions considers the stationary distribution $\pi$ as function of some parameter $\theta$ of $P$. In this case, $\pi$ and $P$ will be denoted by $\pi_\theta$ and $P_\theta$, respectively. Especially, we are interested in obtaining the stationary distribution $\pi_\theta$ as polynomial in parameter $\theta$.

From (2), we get:

$$(I-P_{\bar{\theta}})\ Z_{\bar{\theta}} = I-\Pi_{\bar{\theta}}. \tag{4}$$

Multiplying (4) by $\Pi_\theta$ yields:

$$\Pi_\theta(I-P_{\bar{\theta}})\ Z_{\bar{\theta}} = \Pi_\theta-\Pi_{\bar{\theta}}.$$

This implies that:

$$\Pi_\theta = \Pi_{\bar{\theta}} + \Pi_\theta(I-P_{\bar{\theta}})\ Z_{\bar{\theta}} = \Pi_{\bar{\theta}} + (\underbrace{\Pi_\theta}_{=\Pi_\theta P_\theta} -\Pi_\theta\ P_{\bar{\theta}})\ Z_{\bar{\theta}}.$$

Then, we obtain:

$$\Pi_\theta = \Pi_{\bar{\theta}} + \Pi_\theta(P_\theta-P_{\bar{\theta}})\ Z_{\bar{\theta}}. \tag{5}$$

Inserting (5) into its right side yields:

$$\Pi_\theta = \Pi_{\bar{\theta}} + \Pi_{\bar{\theta}}(P_\theta-P_{\bar{\theta}})\ Z_{\bar{\theta}} + \Pi_\theta[(P_\theta - P_{\bar{\theta}})\ Z_{\bar{\theta}}]^2. \tag{6}$$

Repeating this step $n$ times, we get:

$$\Pi_\theta = \Pi_{\bar{\theta}} \sum_{k=0}^{n} [(P_\theta - P_{\bar{\theta}})\ Z_{\bar{\theta}}]^n + \Pi_\theta[(P_\theta - P_{\bar{\theta}})\ Z_{\bar{\theta}}]^{n+1}. \tag{7}$$

Switching from matrix to vector notation, we obtain the following power series expansion:

$$\pi_\theta = \underbrace{\pi_{\bar{\theta}} \sum_{k=0}^{n} [(P_\theta - P_{\bar{\theta}})Z_{\bar{\theta}}]^k}_{=S(n)} + \underbrace{\pi_\theta\ [(P_\theta - P_{\bar{\theta}})Z_{\bar{\theta}}]^{n+1}}_{=R(n)}. \tag{8}$$

From Equation (8), we can use the series expansion $S(n)$ as polynomial approximation of the stationary distribution $\pi_\theta$, with some error approximation that can be estimated by the remainder term $R(n)$. In words, we obtain:

$$\pi_\theta \approx \pi_{\bar{\theta}} \sum_{k=0}^{n} [(P_\theta - P_{\bar{\theta}})Z_{\bar{\theta}}]^k =: S(n), \tag{9}$$

and

$$R(n) := \pi_\theta[(P_\theta - P_{\bar\theta})Z_{\bar\theta}]^{n+1}. \tag{10}$$

The power series expansion (8) for discrete-time Markov chains on a finite-state space first appeared in Ref.[28]. In this article, we study models the Markov transition kernel of which are linear in the parameter. As we will illustrate with examples in Section 3, this covers an interesting class of problems. Specifically, we assume

$$P_\theta = \theta P_0 + (1-\theta)P_1, \tag{11}$$

with $P_0$ and $P_1$ Markov chains on the same state space. Hence, when $\theta$ is randomized, we get

$$P_{\theta(\omega)} = \theta(\omega)P_0 + (1-\theta(\omega))P_1.$$

Inserting our model for $\theta(\omega)$, we obtain

$$\begin{aligned} P_{\theta(\omega)} &= (\bar\theta + \sigma\varepsilon(\omega))P_0 + (1-\bar\theta-\sigma\varepsilon(\omega))P_1, \\ &= \bar\theta P_0 + (1-\bar\theta)P_1 + \sigma\varepsilon(\omega)(P_0-P_1). \end{aligned}$$

Hence, we can interpret the randomized kernel $P_{\theta(\omega)}$ as a random perturbation of $P_{\bar\theta}$.

In order to propagate input uncertainty, we use the power series expansion (9). For that, inserting $\theta(\omega) = \bar\theta + \sigma\ \varepsilon(\omega)$ together with (11) into (9) and (10) yields the following alternative scheme for approximately computing $\pi_\theta$ and $R(n)$:

$$\pi_{\theta(\omega)} \approx \pi_{\bar\theta} \sum_{k=0}^{n} (\varepsilon(\omega))^k\ [\sigma\ (P_0 - P_1)Z_{\bar\theta}]^k, \tag{12}$$

and

$$R(n,\omega) \approx \pi_\theta[\sigma\ \varepsilon(\omega)\ (P_0 - P_1)Z_{\bar\theta}]^{n+1}. \tag{13}$$

The representation in (12) is called the robust power series expansion of order $n$ for the stationary distribution $\pi_{\theta(\omega)}$, that is, the stationary distribution at $\theta = \theta(\omega)$.

**Remark 2.1.1.** From the uniqueness of the stationary distribution $\pi_{\bar\theta}$, it is easy to see that the fundamental matrix $Z_{\bar\theta}$ exists. However, it is in general unbounded. The boundedness of the fundamental matrix $Z_{\bar\theta}$ depends on the convergence of the power series expansion (3). Provided that $P_{\bar\theta}$ is geometrically ergodic, that is, provided that a finite number $c$ and $\delta \in (0,1)$ exist such that:

$$||P^n_{\bar\theta} - \Pi_{\bar\theta}|| \le c\ \delta^n,$$

for some appropriate norm, then $(I-P_{\bar\theta} + \Pi_{\bar\theta})$ is a bounded matrix and invertible. Therefore, the fundamental matrix $Z_{\bar\theta}$ is bounded; see[17,28] for details.

In order to compute numerically the higher moments of the stationary distribution $\pi_\theta(\omega)$, by considering the uncertainty inflicted on the input parameter $\theta$, one can use the above approach. Particularly, if we assume that the moments $\mathbf{E}[(\varepsilon(\omega))^n]$ are finite, then the expected value and the variance of the stationary distribution $\pi_\theta(\omega)$ can be computed as follows:

$$\mathbf{E}[\pi_{\theta(\omega)}] = \pi_{\bar{\theta}} \sum_{k=0}^{\infty} [\sigma \ (P_0 - P_1) Z_{\bar{\theta}}]^k \mathbf{E}[(\varepsilon(\omega))^k], \tag{14}$$

and

$$\mathbf{Var}[\pi_{\theta(\omega)}] := \mathbf{E}[(\pi_{\theta(\omega)} - \mathbf{E}[\pi_{\theta(\omega)}])^2] = \mathbf{E}[(\pi_{\theta(\omega)})^2] - \mathbf{E}[\pi_{\theta(\omega)}]^2. \tag{15}$$

**Remark 2.1.2.** Using the developed robust power series expansion, one can straightforwardly estimate other characteristics of the stationary distribution $\pi_{\theta(\omega)}$ such as the skewness and the kurtosis coefficient. In fact, the skewness of the random variable $\pi_{\theta(\omega)}$, which is a significant measure of the asymmetry of the probability distribution of $\pi_\theta$ around its mean, can be calculated as follows:

$$\mathbf{Skew}[\pi_{\theta(\omega)}] := \frac{\mathbf{E}[(\pi_{\theta(\omega)} - \mathbf{E}[\pi_{\theta(\omega)}])^3]}{\mathbf{Var}[\pi_{\theta(\omega)}]^{3/2}}. \tag{16}$$

Similarly, the kurtosis coefficient of the stationary distribution $\pi_{\theta(\omega)}$, which is a measure of the tailedness of the probability distribution of the random variable $\pi_{\theta(\omega)}$, can be assessed as follows:

$$\mathbf{Kurt}[\pi_{\theta(\omega)}] := \frac{\mathbf{E}[(\pi_{\theta(\omega)} - \mathbf{E}[\pi_{\theta(\omega)}])^4]}{\left(\mathbf{E}[(\pi_{\theta(\omega)} - \mathbf{E}[\pi_{\theta(\omega)}])^2]\right)^2}. \tag{17}$$

For any cost function $f : \mathbf{Z}^+ \to \mathbf{R}$, one can easily derive the expected value and the variance of the performance measure $\eta_\theta = \pi_\theta f$ of the Markov chain. These quantities of interest are given as follows:

$$\mathbf{E}[\eta_\theta] = \mathbf{E}[\pi_{\theta(\omega)} \ \times \ f] = \sum_{i \geq 0} f(i) \ \times \ \mathbf{E}[\pi_{\theta(\omega)}(i)], \tag{18}$$

and

$$\begin{aligned} \mathbf{Var}[\eta_\theta] = \mathbf{Var}[\pi_{\theta(\omega)} \ \times \ f] &= \sum_{i \geq 0} \left(f(i)\right)^2 \ \times \ \mathbf{Var}[\pi_{\theta(\omega)}(i)] \\ &+ \sum_{i=1} \sum_{j=1, i \neq j} f(i) \ f(j) \ \times \ \mathbf{Cov}(\pi_{\theta(\omega)}(i), \pi_{\theta(\omega)}(j)), \end{aligned} \tag{19}$$

where $\mathbf{E}[\pi_{\theta(\omega)}(i)]$ and $\mathbf{Var}[\pi_{\theta(\omega)}(i)]$ are subsequently approximated by using the formulas (14) and (15), respectively.

**Example 2.1.1.** Choosing the function $f$ as an identity function, that is, $f(i) = i$, we will obtain, by definition, the expected value and the variance of the mean of the Markov chain model $L$. In this case, (18) and (19) become:

$$\mathbf{E}[L_{\theta(\omega)}] = \sum_{i=1}^{N} i \times \mathbf{E}[\pi_{\theta(\omega)}(i)], \tag{20}$$

$$\mathbf{Var}[L_{\theta(\omega)}] = \sum_{i=1}^{N} i^2 \times \mathbf{Var}[\pi_{\theta(\omega)}(i)] + \sum_{i=1}^{N} \sum_{j=1,\, i \neq j}^{N} ij \times \mathbf{Cov}(\pi_{\theta(\omega)}(i), \pi_{\theta(\omega)}(j)). \tag{21}$$

**Example 2.1.2.** One can also choose the function $f$ as an indicator function on $i = k$, that is,

$$f(i) = \mathbf{1}_{\{i=k\}} = \begin{cases} 1, & \text{if} \quad i = k, \\ 0, & \text{otherwise}, \end{cases}$$

where $0 \leq k \leq N$. For this choice, one can derive the desired estimate for the values of the mean and the variance of the probability that $X = k$. Specifically, for $k = N$, we will obtain the estimate of the important measure of performance for a finite buffer queues which is the blocking probability. In other words, we will get:

$$\mathbf{E}[P_b] = \mathbf{E}[P(i = N)] = \mathbf{E}[\pi_{\theta(\omega)}(N)], \tag{22}$$

$$\mathbf{Var}[P_b] = \mathbf{Var}[P(i = N)] = \mathbf{Var}[\pi_{\theta(\omega)}(N)]. \tag{23}$$

In modeling epistemic uncertainty inflicted on parameter values, the important question to be answered is how to choose a specific uncertainty distribution from the possibly incomplete knowledge that is available. If one only knows, for example, the mean $\bar{\theta}$ and the variance $\sigma^2$ of the uncertain parameter $\theta$, and if, in addition, we know that this parameter may take values in $(-\infty, \infty)$, the most general distribution is the normal distribution, denoted by $\mathcal{N}(\bar{\theta}, \sigma^2)$. With "most general distribution", one refers to the fact that this distribution maximizes the entropy. Similarly, if one knows that the uncertain parameter $\theta$ takes values in an interval, say $[a, b]$, then the rectangular distribution on $[a, b]$, denoted by $\mathcal{U}[a, b]$, is the one distribution with maximal entropy. However, there may be statistical knowledge available on the uncertain parameter $\theta$ based on few observed data. In this case, the distribution of $\theta$ is that of the statistic used for estimating uncertain parameter $\theta$.

**Example 2.1.3.** Assume that the random variable $\varepsilon(\omega)$ modeling the epistemic uncertainty distribution in the input parameter $\theta$ follows a standard normal distribution, denoted by $\mathcal{N}(0, 1)$. Then, their $n$-th moments exist and are finite for any non-negative integer $n \geq 1$, we have:

$$\mathbf{E}[(\varepsilon(\omega))^n] = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ (n-1)!!, & \text{if } n \text{ is even,} \end{cases}$$

where $(n-1)!!$ is the semifactorial of $(n-1)$ which is defined as:

$$(n-1)!! = \prod_{l=1}^{s}(2l-1),$$

and $s = \lceil n/2 \rceil$, with $\lceil . \rceil$ is the round up function.

Thereby, for an even positive integer $n = 2p, \quad p \geq 0$, the expected value of $\pi_\theta(\omega)$ given in (14), can be obtained in more explicit form as follows:

$$\begin{aligned} \mathbf{E}[\pi_{\theta(\omega)}] &= \pi_{\bar{\theta}} \sum_{n=0}^{\infty} [\sigma \;\; (P_0 - P_1) Z_{\bar{\theta}}]^n \;\; \mathbf{E}[(\varepsilon(\omega))^n] \\ &= \pi_{\bar{\theta}} \sum_{p=0}^{\infty} [\sigma \;\; (P_0 - P_1) Z_{\bar{\theta}}]^{2p} (2p-1)!! \end{aligned} \tag{24}$$

In the same vein, the variance of $\pi_i(\omega)$ can be derived as follows:

$$\begin{aligned} \mathbf{Var}[\pi_{\theta(\omega)}(i)] &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \alpha_{i,n} \;\; \alpha_{i,m} \mathbf{Cov}(\varepsilon(\omega)^n, \varepsilon(\omega)^m) \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \alpha_{i,n} \;\; \alpha_{i,m} [\mathbf{E}(\varepsilon(\omega)^{n+m}) - \mathbf{E}(\varepsilon(\omega)^n) \;\; \mathbf{E}(\varepsilon(\omega)^m)] \\ &= \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \alpha_{i,2p} \;\; \alpha_{i,2q} [\mathbf{E}(\varepsilon(\omega)^{2p+2q}) - \mathbf{E}(\varepsilon(\omega))^{2p} \;\; \mathbf{E}(\varepsilon(\omega))^{2q}] \\ &\quad + \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \alpha_{i,2p+1} \;\; \alpha_{i,2q+1} \mathbf{E}(\varepsilon(\omega)^{2p+2q+2}) \\ &= \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \alpha_{i,2p}\alpha_{i,2q} [(2p+2q-1)!! - (2p-1)!!(2q-1)!!] \\ &\quad + \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \alpha_{i,2p+1} \;\; \alpha_{i,2q+1} (2p+2q+1)!! \end{aligned} \tag{25}$$

where

$$\alpha_{i,k} = \pi_{\bar{\theta}} [\sigma \;\; (P_0 - P_1) Z_{\bar{\theta}}]^k e_i^T, \tag{26}$$

with $e_i^T = (0\ldots, \underbrace{1}_{at \;\; the \;\; i \;\; position}, \ldots 0)$.

**Example 2.1.4.** Suppose that the parameter $\theta$ takes values in $[a, b]$, then the rectangular distribution on $[a, b]$, denoted by $\mathcal{U}[a, b]$, is the natural candidate to model the uncertainty distribution of the parameter $\theta$. Especially, assume that the random variable $\varepsilon(\omega)$ is $\mathcal{U}[-1, 1]$. In this case, their $n$-th moments exist and are finite, and we have:

$$\mathbf{E}[(\varepsilon(\omega))^n] = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \dfrac{1}{n+1}, & \text{if } n \text{ is even.} \end{cases}$$

Therefore, the mean and the variance of the stationary distribution $\pi_\theta(\omega)$ are given, respectively, as follows:

$$\begin{aligned} \mathbf{E}[\pi_{\theta(\omega)}] &= \sum_{n=0}^{\infty} [\sigma\ (P_0 - P_1) Z_{\bar{\theta}}]^n\ \mathbf{E}[(\varepsilon(\omega))^n], \\ &= \sum_{p=0}^{\infty} [\sigma\ (P_0 - P_1) Z_{\bar{\theta}}]^{2p}\ \frac{1}{2p+1}. \end{aligned} \tag{27}$$

$$\begin{aligned} \mathbf{Var}[\pi_{\theta(\omega)}(i)] &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \alpha_{i,n}\ \alpha_{i,m} \mathbf{Cov}(\varepsilon(\omega)^n, \varepsilon(\omega)^m), \\ &= \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \alpha_{i,2p}\ \alpha_{i,2q} \left[\frac{1}{2p+2q+1} - \frac{1}{(2p+1)(2q+1)}\right] \\ &+ \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \alpha_{i,2p+1}\ \alpha_{i,2q+1} \frac{1}{2p+2q+3}, \end{aligned} \tag{28}$$

where $\alpha_{i,k}$ is given in (26).

The alternative computational approach to characterize, in a complete and univocal manner, the uncertain stationary distribution $\pi_\theta(\omega)$ is to estimate its probability density function. Then, using the polynomial $S_i$ obtained in (12), the probability density function of $\pi_i$ can be accurately computed by using the well-known random variable-transformation method stated as follows[31]:

$$\mathbf{f}_{\pi_i}(x) = \sum_{j=1}^{r} \frac{f_\varepsilon(x)}{|S_j'(x)|}, \tag{29}$$

where $f_\varepsilon$ is the probability density function of the random variable $\varepsilon$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r$ are the $r$ $(\leq n)$ real roots of the polynomial equation $\pi_i := S_i(\varepsilon)$, and $S_j'$ is the first derivative of the polynomial $S_j$ with respect to $\varepsilon_j$:

$$S_j'(\varepsilon_j) := \left.\frac{dS_j}{d\varepsilon}\right|_{\varepsilon=\varepsilon_j} \neq 0. \tag{30}$$

### 2.2. Convergence of the power series expansions

In this section, we investigate the convergence of the power series expansions outlined above. Specifically, we require sufficient conditions such that the series expansion (12) tends to the stationary distribution $\pi_\theta(\omega)$ as $n$ tends to infinity. Such conditions are expressed in terms of the fundamental matrix. The precise statement is presented in the following theorem.

**Theorem 2.2.1.** *Suppose that the moments* $\mathbf{E}[(\varepsilon(\omega))^n]$ *are finite. Then, there exists a finite constant* $\kappa$ *such that* $\mathbf{E}[(\varepsilon(\omega))^n] \leq \kappa^n$, *and we assume that a finite constant* $c$ *exists such that:*

$(\mathbf{C})\|(P_0 - P_1)Z_{\bar{\theta}}\| \leq c < 1/\kappa\ \sigma.$

*Then the power series expansion (12) converges in mean towards to the stationary distribution* $\pi_\theta(\omega)$. *Moreover, a lower bound for the radius of convergence of the power series is given by:*

$$\rho > \kappa\ \sigma, \tag{31}$$

*where* $\sigma$ *is the standard deviation of the uncertain parameter* $\theta(\omega)$.

*Proof.* We need to show that:

$$\lim_{n\to\infty} \mathbf{E}\|\pi_\theta(\omega)\ [(P_\theta(\omega) - P_{\bar{\theta}})Z_{\bar{\theta}}]^{n+1}\| = \lim_{n\to\infty} \mathbf{E}\left\|\pi_{\bar{\theta}} \sum_{k=n+1}^{\infty} [(P_\theta(\omega) - P_{\bar{\theta}})Z_{\bar{\theta}}]^k\right\| = 0. \tag{32}$$

Notice that the above limits are often required to be unique in an appropriate sense.

Assume that condition $(\mathbf{C})$ holds and the moments $\mathbf{E}[(\varepsilon(\omega))^n]$ are finite, then the constant $c$ is smaller than $1/\kappa\ \sigma$. So taking the largest possible value and letting $c$ tend to $1/\kappa\ \sigma$, we arrive at:

$$\lim_{n\to\infty} \mathbf{E}\left\|\pi_{\bar{\theta}} \sum_{k=n+1}^{\infty} [(P_\theta(\omega) - P_{\bar{\theta}})Z_{\bar{\theta}}]^k\right\| = 0. \tag{33}$$

Furthermore, we will mainly focus on obtaining a lower bound to radius of convergence. Let $\rho$ be the radius of convergence of the power series expansions (12). It is defined as the radius of the largest disk in which the series converges. Using the Cauchy-Hadamard formula, it holds that:

$$\begin{aligned} 1/\rho &= \limsup_{k\to\infty} \|\pi_{\bar{\theta}}[(P_0 - P_1)Z_{\bar{\theta}}]^k\|^{\frac{1}{k}}, \\ &\leq \|\pi_{\bar{\theta}}[(P_0 - P_1)Z_{\bar{\theta}}]\|, \\ &\leq \|(P_0 - P_1)Z_{\bar{\theta}}\|. \end{aligned} \tag{34}$$

This together with condition **(C)** allows to show that:

$$\|(P_0 - P_1)Z_{\bar{\theta}}\| < 1/\kappa\ \sigma. \tag{35}$$

Therefore, we get:

$$\rho > \kappa\ \sigma. \tag{36}$$

This means that $1/\kappa\ \sigma$ is the maximal value of the uncertainty values in mean inflicted on the parameter $\theta(\omega)$.

### *2.3. A bound for the remainder term of the power series expansions*

It is interesting from a computational point of view to estimate the remainder term (13) and to bound it. In the sequel, we obtain an upper bound for the expectation of the remainder term. This estimate will be used to determinate the approximation order.

**Theorem 2.3.1.** *Suppose that conditions of Theorem 2.2.1. hold, then an upper bound for the expected value of the remainder term is given by:*

$$\mathbf{E}\left[\sum_{k=n+1}^{\infty} \|\sigma\ \varepsilon(\omega)[(P_0 - P_1)Z_{\bar{\theta}}]\|^k\right] \leq \frac{(\kappa\ \sigma\ c)^{n+1}}{1 - \kappa\ \sigma\ c}. \tag{37}$$

*Proof.* The power series expansion introduced above in (8) can be rewritten as:

$$\pi_\theta(\omega) = \pi_{\bar{\theta}} \sum_{k=0}^{\infty} [(P_\theta(\omega) - P_{\bar{\theta}})Z_{\bar{\theta}}]^k. \tag{38}$$

In the same vein, the remainder term (10) can be reexpressed as follows:

$$R(n,\omega) = \pi_{\bar{\theta}} \sum_{k=n+1}^{\infty} [(P_\theta(\omega) - P_{\bar{\theta}})Z_{\bar{\theta}}]^k. \tag{39}$$

Taking 1-norm of the remainder term (39) yields:

$$\left\|\pi_{\bar{\theta}} \sum_{k=n+1}^{\infty} [(P_\theta(\omega) - P_{\bar{\theta}})Z_{\bar{\theta}}]^k\right\| \leq ||\pi_{\bar{\theta}}|| \sum_{k=n+1}^{\infty} \|\sigma\ \varepsilon(\omega)[(P_0 - P_1)Z_{\bar{\theta}}]\|^k, \tag{40}$$

$$\leq \sum_{k=n+1}^{\infty} \|\sigma\ \varepsilon(\omega)[(P_0 - P_1)Z_{\bar{\theta}}]\|^k. \tag{41}$$

Then, the expected value of the 1-norm of the remainder term can be estimated as follows:

$$\mathbf{E}\left[\sum_{k=n+1}^{\infty}\|\sigma\ \varepsilon(\omega)[(P_0-P_1)Z_{\bar{\theta}}]\|^k\right] \le \sum_{k=n+1}^{\infty}\mathbf{E}[(\varepsilon(\omega))^k]\|\sigma\ [(P_0-P_1)Z_{\bar{\theta}}]\|^k, \\ \le \sum_{k=n+1}^{\infty}\kappa^k\|\sigma\ [(P_0-P_1)Z_{\bar{\theta}}]\|^k, \\ \le \sum_{k=n+1}^{\infty}\|\kappa\ \sigma\ [(P_0-P_1)Z_{\bar{\theta}}]\|^k. \tag{42}$$

Using key condition **(C)** allows us on the one hand to guarantee that the norm of the remainder decreases at a geometric rate towards zero as $n$ tends to infinity, on the other hand to obtain the following upper bound for the remainder term:

$$\mathbf{E}\left[\sum_{k=n+1}^{\infty}\|\sigma\ \varepsilon(\omega)[(P_0-P_1)Z_{\bar{\theta}}]\|^k\right] \le \sum_{k=n+1}^{\infty}(\kappa\ \sigma\ c)^k, \\ \le \frac{(\kappa\ \sigma\ c)^{n+1}}{1-\kappa\ \sigma\ c}. \tag{43}$$

For a given precision $\xi > 0$, one can use (43) to compute numerically the order of approximation $n$, then via the power series expansions, we derive the desired estimate for the values of higher moments of the stationary distribution $\pi_\theta$.

**Remark 2.3.1.** It is worth noting that computing the constant $\kappa$ evoked in the supposition of Theorem 2.3.1., can be made according to the chosen probability distribution of the random variable $\varepsilon(\omega)$. For example, if we assume that $\varepsilon(\omega) \sim \mathcal{N}(0,1)$, then we can easily find $\kappa = 2$. In the same vein, if $\varepsilon(\omega) \sim \mathcal{U}[-1,1]$, one obtains $\kappa = 1$.

### *2.4. The robust power series expansion algorithm*

In this section, we propose a robust power series expansion algorithm which allows us to compute the uncertain stationary distribution $\pi_{\theta(\omega)}$, with assumption that the parameter $\theta(\omega)$ is modeled as random variable representing the uncertainties inflicted on this parameter. The following algorithm summarizes the main steps of the proposed approach outlined above.

**Step 0. Introduce**

the expected value of the input parameter: $\bar{\theta}$, the standard deviation of the input parameter: $\sigma$,

the precision: $\xi > 0$;

**Step 1. Choose** a particular epistemic uncertainty distribution associated to the random variable $\varepsilon(\omega)$;
**Step 2. Check** the convergence condition **(C)**;
**Step 3. Estimate** the upper bound for the remainder term from (37);
**Step 4. Determine** the power series expansions order $n$ such that:

$$\frac{(\kappa\ \sigma\ c)^{n+1}}{1-\kappa\ \sigma\ c} \leq \xi;$$

**Step 5. Compute** the main objects related to the nominal model without uncertainty involving in computing the elements of the power series expansions:

the stationary distribution: $\pi_{\bar{\theta}}$,

the ergodic matrix: $\Pi_{\bar{\theta}}$,

the fundamental matrix: $Z_{\bar{\theta}}$;

**Step 6. Approximate** the main quantities of interest:

the expected value of $\pi_{\theta(\omega)}$ from (14): $\mathbf{E}[\pi_{\theta(\omega)}]$,

the variance of $\pi_{\theta(\omega)}$ from (15): $\mathbf{Var}[\pi_{\theta(\omega)}]$,

the probability density function of $\pi_{\theta(\omega)}$ from (29): $\mathbf{f}_{\pi_i}(x)$,

the expected value of the performance measure $\eta_\theta$ from (18): $\mathbf{E}[\eta_{\theta(\omega)}]$,

the variance of the performance measure $\eta_{\theta(\omega)}$ from (19): $\mathbf{Var}[\eta_{\theta(\omega)}]$.

**Algorithm 1:** "Robust Computational Algorithm"

In the sequel, we provide the main steps of Monte Carlo simulation which allow us to understand thoroughly the procedure of obtaining the same quantities of interest.

**Step 0. Introduce** the inputs

the expected value of the input parameter: $\bar{\theta}$,

the standard deviation of the input parameter: $\sigma$;

the size sample: $N_{MC}$,

the number of replications: $m$.

**Step 1. Generate** a sample for the random variable $\varepsilon(\omega)$ of size $n$ following the chosen epistemic uncertainty distribution;
**Step 2. Compute** the transition matrix $P_\theta(\omega)$ at each point of the sample associated with $\varepsilon(\omega)$;
**Step 3. Compute** the stationary distribution $\pi_{\theta(\omega)}$ at each point of the sample associated with $\varepsilon(\omega)$;
**Step 4. Calculate** the main quantities of interest at each point of the sample associated with $\varepsilon(\omega)$;
**Step 5. Repeat** $m$ times the procedures of Steps 1–4; then calculate the quantities of interest of the induced results.

**Algorithm 2:** "Simulation Algorithm"

**Remark 2.4.1.** Note that the input elements of the initial step, that is, the mean $\bar{\theta}$ and the standard deviation $\sigma$, associated to the input parameter $\theta(\omega)$, can be computed by using any standard statistical technique. Under certain circumstances, however, the mean tends to be less useful than the median. For example, when we have an extreme values, the median is the measure which is able to make much more complete use of the available data. In such situations, the median resumes the essential information on data than the mean. Thereby, the measure which should be used depends on the nature of available data itself.

## 3. Robust analysis of queueing models with breakdowns and repairs

In this section, we illustrate the power series expansions outlined above to numerically approximate the performance measures of queueing models with breakdowns and repairs, by integrating of parameter uncertainty. Particularly, we present two worked out queueing examples: the M/G/1/N and the M/M/c/N queues with breakdowns and repairs, where we consider that the probability of a server breakdown is not known with certainty. More formally, let $\theta$ denote the probability of a server breakdown at the beginning of a service; then, robust perturbation analysis seeks to compute the stationary distribution of the queue-length process, denoted by $\pi_\theta$, and other performances under the assumption that the probability $\theta$ is uncertain. Indeed, data on the period characterizing the interval of time between occurrence of breakdowns of the server is only available in censored form and thereby estimating mean times of such periods is a statistical challenge. For this reason, we are typically confronted with uncertainty concerning the true value of the parameters defining the distributions of the time between breakdowns, in our case, the probability of a server breakdown $\theta$. Parametric uncertainty analysis of the M/G/1/N queue with breakdowns and repairs is provided in Section 3.1, and that of the M/M/c/N queue with breakdowns in Section 3.2.

### 3.1. Analysis of the M/G/1/N queue with breakdowns and repairs

Consider a M/G/1/N queue with server's breakdowns and repairs. Customers arrive according to a Poisson process with rate $\lambda$. Arriving customers that find the queue full are assumed to be blocked and lost. The service time provided by a single server is independent and identically distributed random variable with a general distribution function $S(x)$, having finite mean $1/\mu$. The server can serve only one customer at a time. Customers are served according to the first come first served (FCFS) discipline. In this model, we consider the breakdowns with losses. If the server is broken down, then the blocked customer may leave the system altogether without being served. Assume that with probability $\theta$, the server is operational and serves a customer, otherwise the server is subject to random breakdowns, which can occur only at the beginning of each service with a probability $(1-\theta)$. In this case, the server is sent for repair and it cannot be occupied, the length of which is exponentially distributed with a distribution function $R(x)$ with rate $r$. Immediately after that the server is repaired, the customer who has just begun its service is lost with a probability $(1-\gamma)$, otherwise is recovered with a probability $\gamma$. The service, breakdown, and repair processes are independent of each other.

We denote by $X_n$ the number of customers in the system immediately after the service completions and completion of a repair. The sequence of random variables $X = \{X_n : n \in \mathbf{N}\}$ constitutes a Markov chain, and has state space $S = \{0, 1, ..., N-1\}$. Let $P_{\theta,\gamma}$ denote the state-transition probability matrix of the embedded Markov chain $X$, then $P_1$ represents the system with no breakdowns whereas $P_0$ models the system with certain server breakdown. Furthermore, after the repair of the server, $P_0^{(1)}$ also models the system without breakdowns while $P_0^{(0)}$ describes the system with server breakdown. Due to our assumption that breakdowns occur independently of everything else, it holds that:

$$P_{\theta,\gamma} = \theta\ P_1 + (1-\theta)\Big(\gamma\ P_0^{(1)} + (1-\gamma)\ P_0^{(0)}\Big), \quad \text{for} \quad \theta, \gamma \in [0, 1]. \tag{44}$$

Note that after the repair period, we will have another service period and/or another repair period with the same distributions as the first ones, which means that $P_1 = P_0^{(1)}$ and $P_0 = P_0^{(0)}$. Then we arrive at:

$$P_{\theta,\gamma} = \Big(\theta + (1-\theta)\ \gamma\Big)\ P_1 + (1-\theta)\ (1-\gamma)\ P_0, \tag{45}$$

where

$$P_1 = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{N-2} & 1-\sum_{k=0}^{N-2}\alpha_k \\ \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{N-2} & 1-\sum_{k=0}^{N-2}\alpha_k \\ 0 & \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{N-3} & 1-\sum_{k=0}^{N-3}\alpha_k \\ 0 & 0 & \alpha_0 & \alpha_1 & \cdots & \alpha_{N-4} & 1-\sum_{k=0}^{N-4}\alpha_k \\ & & \ddots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha_0 & 1-\alpha_0 \end{pmatrix}, P_0 = \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_{N-2} & 1-\sum_{k=0}^{N-2}\beta_k \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_{N-2} & 1-\sum_{k=0}^{N-2}\beta_k \\ 0 & \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{N-3} & 1-\sum_{k=0}^{N-3}\beta_k \\ 0 & 0 & \beta_0 & \beta_1 & \cdots & \beta_{N-4} & 1-\sum_{k=0}^{N-4}\beta_k \\ & & \ddots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \beta_0 & 1-\beta_0 \end{pmatrix}, \tag{46}$$

with

$$\alpha_k = \int_0^\infty e^{-\lambda x}\frac{(\lambda x)^k}{k!}dS(x), \quad k = 0, \ldots, N-2, \tag{47}$$

$$\beta_k = \int_0^\infty e^{-\lambda x}\frac{(\lambda x)^k}{k!}dR(x), \tag{48}$$

$$= \frac{r}{r+\lambda}\left(\frac{\lambda}{\lambda+r}\right)^k, \quad k = 0, \ldots, N-2. \tag{49}$$

In the following, we apply Algorithm 1 to the M/G/1/N queue with breakdowns and repairs with the purpose of computing some quantities of interest and/or estimating the probability density functions of the uncertain stationary distribution, where we consider that the probability of a server breakdown $\theta$ is uncertain. For that, we consider the new model introduced in (1) for the probability of a server breakdown:

$$\theta(\omega) = \bar{\theta} + \sigma\ \varepsilon(\omega),$$

with mean $\bar{\theta} = 0.5$ and standard deviation $\sigma = 0.09$. Moreover, we allow the random variable $\varepsilon(\omega)$ to follow two different distributions: the standard normal distribution and the rectangular distribution on $[-1, 1]$. For the numerical experiment, we fix the model parameters as follows: the buffer space $N=6$, the arrival rate $\lambda=7$, the repair rate $r = 2.5$, and the lost probability $\gamma = 0.2$. If we now let, for example, $\varepsilon(\omega) \sim \mathcal{N}(0, 1)$, then the histogram and plot corresponding to the probability of a server breakdown (1) are shown in Figure 1.

The second step of Algorithm 1 consists to check the convergence condition **(C)**. Therefore, for $\varepsilon(\omega) \sim \mathcal{N}(0, 1)$ condition **(C)** becomes:

$$\|(P_0 - P_1)Z_{\bar{\theta}}\|_1 \le c < 48.2253086,$$

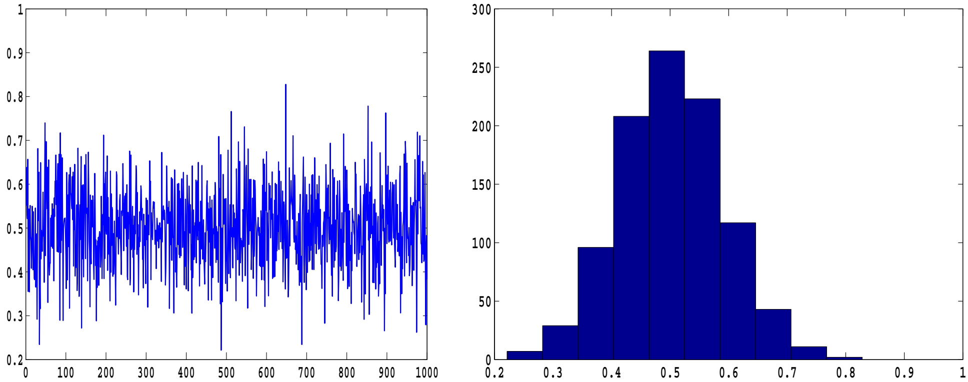

**Figure 1.** Histogram and plot for the probability of a server breakdown $\theta$ with $\bar{\theta} = 0.5$ and $\sigma = 0.09$.

whereas for the case $\varepsilon(\omega) \sim \mathcal{U}[-1, 1]$, this condition reads:

$$\|(P_0 - P_1)Z_{\bar{\theta}}\|_1 \le c < 1.929012346.$$

Throughout our numerical analysis, four types of service time distributions are considered, namely Exponential (M) with parameter $\mu = 4$, Deterministic (D) with mean $d = 1/4$, Weibull (Weibull) having a density function of the form:

$$s(x) = \mu\ \delta\ (\mu\ x)^{\delta-1}\ e^{-(\mu x)^{\delta}},\ \ x \ge 0,$$

where we let $\mu = 4$ and $\delta = 2$, and Hyperexponential ($H_2$) with density function:

$$s(x) = q\ \mu_1\ e^{-\mu_1 x} + (1-q)\mu_2\ e^{-\mu_2 x},\ \ x \ge 0,$$

where $\mu_1 = 4$ and $\mu_2 = 3.5$ and $q = 0.3$.

By applying Algorithm 1 to the M/G/1/6 queue with breakdowns and repair, one can approximate the expected value and the variance of stationary distribution $\pi(\omega)$. We set the desired precision value $\xi = 10^{-5}$, and the behavior of the upper bound for the expected value of the remainder term $\mathbf{E}[R(n, \varepsilon(\omega))]$ with respect to changes of the power series expansions order $n$ as shown in Figure 2.

As can be seen from Figure 2, for the specified distributions of the service time, we obtain the different values for power series expansions of order $n$: for the M/D/1/6 queue with breakdowns and repairs, we get $n = 6$ and for the other queues M/M/1/6, M/$H_2$/1/6, and M/Weibull/1/6 with breakdowns and repairs, we get $n = 4$.

Tables 1–8 show the computational results, for the four studied queues, related to the obtained expected value and variance, respectively, of each component $\pi_i$, $i = 0, 1, \ldots, 6$ computed by power series expansions and by using Monte Carlo simulations.

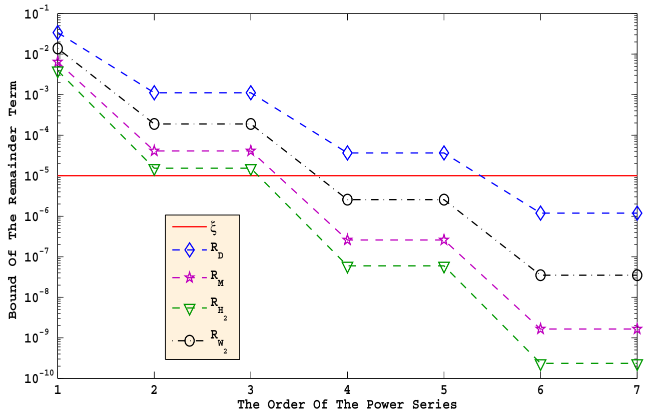


**Figure 2.** Bound of the remainder term.

**Table 1.** Expected value of the steady-state vector in M/M/1/6 queue.

| Expected value | $\pi_{\theta(\omega)}(i)$ | PSE | MC ($N_{MC} = 100000$) |
|---|---|---|---|
| $\varepsilon \sim \mathcal{N}(0,1)$ | $\pi_0$ | 0.0130225 | 0.0130219 |
| | $\pi_1$ | 0.0271354 | 0.0271344 |
| | $\pi_2$ | 0.0568966 | 0.0568948 |
| | $\pi_3$ | 0.1196329 | 0.1196305 |
| | $\pi_4$ | 0.2519628 | 0.2519615 |
| | $\pi_5$ | 0.5313495 | 0.5313566 |
| $\varepsilon \sim \mathcal{U}[-1,1]$ | $\pi_0$ | 0.0129488 | 0.0129482 |
| | $\pi_1$ | 0.0270507 | 0.0270498 |
| | $\pi_2$ | 0.0568277 | 0.0568263 |
| | $\pi_3$ | 0.1196320 | 0.1196304 |
| | $\pi_4$ | 0.2520805 | 0.2520800 |
| | $\pi_5$ | 0.5314600 | 0.5314650 |

**Table 2.** Variance of the steady-state vector in M/M/1/6 queue.

| Variance | $\pi_{\theta(\omega)}(i)$ | PSE | MC ($N_{MC} = 100000$) |
|---|---|---|---|
| $\varepsilon \sim \mathcal{N}(0,1)\ \times 10^{-3}$ | $\pi_0$ | 0.0032570 | 0.0032662 |
| | $\pi_1$ | 0.0081667 | 0.0081907 |
| | $\pi_2$ | 0.0167521 | 0.0168049 |
| | $\pi_3$ | 0.0213939 | 0.0214674 |
| | $\pi_4$ | 0.0019509 | 0.0019601 |
| | $\pi_5$ | 0.2173271 | 0.2180464 |
| $\varepsilon \sim \mathcal{U}[-1,1] \times 10^{-4}$ | $\pi_0$ | 0.0106395 | 0.0106734 |
| | $\pi_1$ | 0.0269416 | 0.0269896 |
| | $\pi_2$ | 0.0557004 | 0.0557353 |
| | $\pi_3$ | 0.0715113 | 0.0714925 |
| | $\pi_4$ | 0.0064184 | 0.0063994 |
| | $\pi_5$ | 0.7234414 | 0.7238343 |

From the numerical results, obtained by applying Algorithm 1 and Monte Carlo simulation to the four considered queues, it is easy to see a very good match between the power series approximation and the simulated one.

We now turn to obtaining the probability density functions (pdfs) associated with the components of the stationary distribution. For this, we assume that the random variable $\varepsilon(\omega)$ follows a rectangular distribution on $[-1,1]$. The obtained results, for the four considered queues, are illustrated graphically in Figures 3–6.

**Table 3.** Expected value of the steady-state vector in M/D/1/6 queue.

| Expected value | $\pi_{\theta(\omega)}(i)$ | PSE | MC ($N_{MC} = 100000$) |
|---|---|---|---|
| $\varepsilon \sim \mathcal{N}(0,1)$ | $\pi_0$ | 0.0000260 | 0.0000259 |
| | $\pi_1$ | 0.0001925 | 0.0001926 |
| | $\pi_2$ | 0.0015018 | 0.0015036 |
| | $\pi_3$ | 0.0121356 | 0.0121508 |
| | $\pi_4$ | 0.1016168 | 0.1017020 |
| | $\pi_5$ | 0.8845271 | 0.8844248 |
| $\varepsilon \sim \mathcal{U}[-1,1]$ | $\pi_0$ | 0.0129488 | 0.0129482 |
| | $\pi_1$ | 0.0270507 | 0.0270498 |
| | $\pi_2$ | 0.0568277 | 0.0568263 |
| | $\pi_3$ | 0.1196320 | 0.1196304 |
| | $\pi_4$ | 0.2520805 | 0.2520800 |
| | $\pi_5$ | 0.5314600 | 0.5314650 |

**Table 4.** Variance of the steady-state vector in M/D/1/6 queue.

| Variance | $\pi_{\theta(\omega)}(i)$ | PSE | MC ($N_{MC} = 100000$) |
|---|---|---|---|
| $\varepsilon \sim \mathcal{N}(0,1)$ $\times 10^{-3}$ | $\pi_0$ | 0.0000006 | 0.0000006 |
| | $\pi_1$ | 0.0000209 | 0.0000214 |
| | $\pi_2$ | 0.0006949 | 0.0006904 |
| | $\pi_3$ | 0.0193714 | 0.0191301 |
| | $\pi_4$ | 0.3079438 | 0.3041142 |
| | $\pi_5$ | 0.5234175 | 0.5169023 |
| $\varepsilon \sim \mathcal{U}[-1,1] \times 10^{-3}$ | $\pi_0$ | 0.0000001 | 0.0000001 |
| | $\pi_1$ | 0.0000051 | 0.0000051 |
| | $\pi_2$ | 0.0002020 | 0.0002020 |
| | $\pi_3$ | 0.0063122 | 0.0063140 |
| | $\pi_4$ | 0.1028191 | 0.1028836 |
| | $\pi_5$ | 0.1736684 | 0.1737620 |

**Table 5.** Expected value of the steady-state vector in M/Weibull/1/6 queue.

| Expected value | $\pi_{\theta(\omega)}(i)$ | PSE | MC ($N_{MC} = 100000$) |
|---|---|---|---|
| $\varepsilon \sim \mathcal{N}(0,1)$ | $\pi_0$ | 0.0016110 | 0.0016096 |
| | $\pi_1$ | 0.0042896 | 0.0042857 |
| | $\pi_2$ | 0.0100279 | 0.0100164 |
| | $\pi_3$ | 0.0303470 | 0.0303151 |
| | $\pi_4$ | 0.1345959 | 0.1345279 |
| | $\pi_5$ | 0.8191283 | 0.8192451 |
| $\varepsilon \sim \mathcal{U}[-1,1]$ | $\pi_0$ | 0.0016079 | 0.0016078 |
| | $\pi_1$ | 0.0042788 | 0.0042785 |
| | $\pi_2$ | 0.0099792 | 0.0099785 |
| | $\pi_3$ | 0.0302314 | 0.0302294 |
| | $\pi_4$ | 0.1346939 | 0.1346888 |
| | $\pi_5$ | 0.8192085 | 0.8192167 |

It is worth noting that one can easily calculate the expected value and variance of the stationary distribution $\pi(\theta(\omega))$ by using the following expressions:

$$\mathbf{E}[\pi_i(\theta(\omega))] \approx \int_0^1 y f_{\pi_i}(y) dy, \tag{50}$$

and

**Table 6.** Variance of the steady-state vector in M/Weibull/1/6 queue.

| Variance | $\pi_{\theta(\omega)}(i)$ | PSE | MC ($N_{MC} = 100000$) |
|---|---|---|---|
| $\varepsilon \sim \mathcal{N}(0,1)\ \times 10^{-3}$ | $\pi_0$ | 0.0000517 | 0.0000502 |
| | $\pi_1$ | 0.0003985 | 0.0003874 |
| | $\pi_2$ | 0.0029061 | 0.0028219 |
| | $\pi_3$ | 0.0237697 | 0.0230933 |
| | $\pi_4$ | 0.1631204 | 0.1589700 |
| | $\pi_5$ | 0.4080360 | 0.3972059 |
| $\varepsilon \sim \mathcal{U}[-1,1] \times 10^{-3}$ | $\pi_0$ | 0.0000172 | 0.0000172 |
| | $\pi_1$ | 0.0001330 | 0.0001329 |
| | $\pi_2$ | 0.0009647 | 0.0009639 |
| | $\pi_3$ | 0.0079132 | 0.0079069 |
| | $\pi_4$ | 0.0543971 | 0.0543524 |
| | $\pi_5$ | 0.1360963 | 0.1359862 |

**Table 7.** Expected value of the steady-state vector in $M/H_2/1/N$ queue.

| Expected value | $\pi_{\theta(\omega)}(i)$ | PSE | MC ($N_{MC} = 100000$) |
|---|---|---|---|
| $\varepsilon \sim \mathcal{N}(0,1)$ | $\pi_0$ | 0.0102246 | 0.0102231 |
| | $\pi_1$ | 0.0226247 | 0.0226224 |
| | $\pi_2$ | 0.0502660 | 0.0502627 |
| | $\pi_3$ | 0.1118736 | 0.1118699 |
| | $\pi_4$ | 0.2492480 | 0.2492470 |
| | $\pi_5$ | 0.5557627 | 0.5557746 |
| $\varepsilon \sim \mathcal{U}[-1,1]$ | $\pi_0$ | 0.0101858 | 0.0101840 |
| | $\pi_1$ | 0.0225769 | 0.0225737 |
| | $\pi_2$ | 0.0502231 | 0.0502182 |
| | $\pi_3$ | 0.1118674 | 0.1118615 |
| | $\pi_4$ | 0.2493155 | 0.2493131 |
| | $\pi_5$ | 0.5558310 | 0.5558491 |

**Table 8.** Variance of the steady-state vector in $M/H_2/1/N$ queue.

| Variance | $\pi_{\theta(\omega)}(i)$ | PSE | MC ($N_{MC} = 100000$) |
|---|---|---|---|
| $\varepsilon \sim \mathcal{N}(0,1)\ \times 10^{-3}$ | $\pi_0$ | 0.0013427 | 0.0013348 |
| | $\pi_1$ | 0.0038361 | 0.0038138 |
| | $\pi_2$ | 0.0090223 | 0.0089705 |
| | $\pi_3$ | 0.0136265 | 0.0135493 |
| | $\pi_4$ | 0.0024620 | 0.0024483 |
| | $\pi_5$ | 0.1292548 | 0.1285180 |
| $\varepsilon \sim \mathcal{U}[-1,1]\ \times 10^{-4}$ | $\pi_0$ | 0.0044155 | 0.0044131 |
| | $\pi_1$ | 0.0126970 | 0.0126782 |
| | $\pi_2$ | 0.0300154 | 0.0299481 |
| | $\pi_3$ | 0.0454855 | 0.0453574 |
| | $\pi_4$ | 0.0081975 | 0.0081652 |
| | $\pi_5$ | 0.4304644 | 0.4294469 |

$$\mathbf{Var}[\pi_i(\theta(\omega))] \approx \int_0^1 y^2 f_{\pi_i}(y)dy - \left(\int_0^1 y f_{\pi_i}(y)dy\right)^2, \tag{51}$$

where $f_{\pi_i}$ is the pdf of the random variable $\pi_i(\omega)$.

### 3.2. Analysis of the M/M/c/N queue with breakdowns

Consider a multi-server queue with $c$ servers. Costumers arrive to the queue according to a Poisson $\lambda$ arrival stream. There are $c \geq 1$ servers with

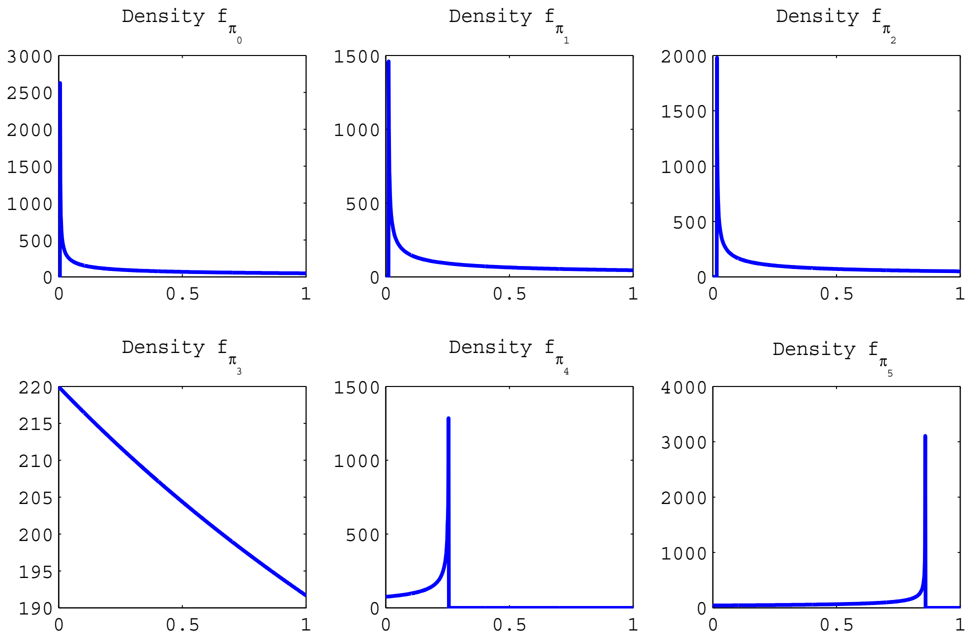


**Figure 3.** Probability density function of stationary distribution of the M/M/1/6 queue with breakdowns and repairs.

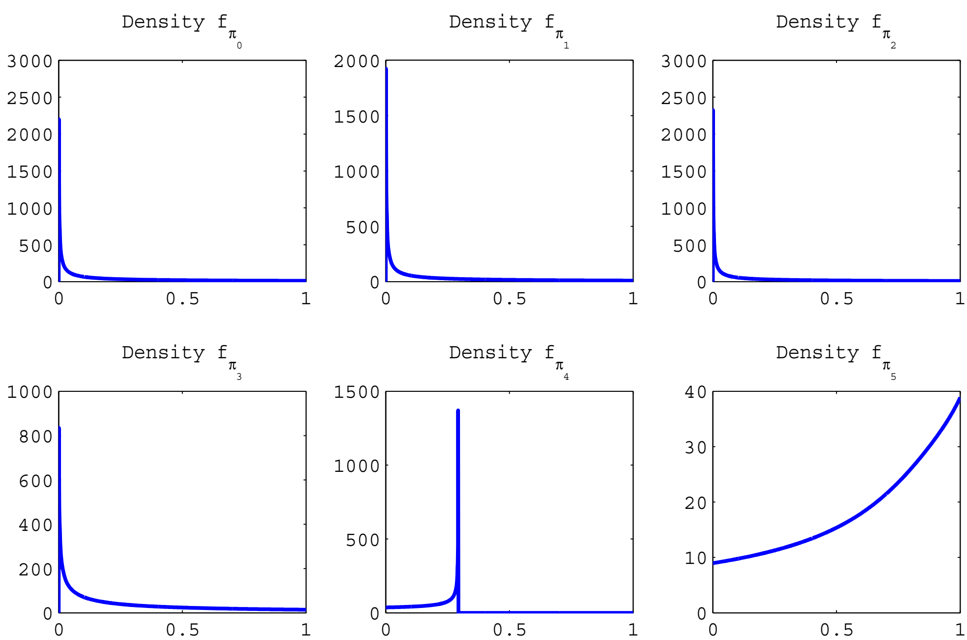


**Figure 4.** Probability density function of stationary distribution of the M/D/1/6 queue with breakdowns and repairs.

exponential distributed service times with rate $\mu$. If a server becomes available and there is a customer in queue waiting for service, the server may experience a breakdown with probability $\theta$. We assume that the probability

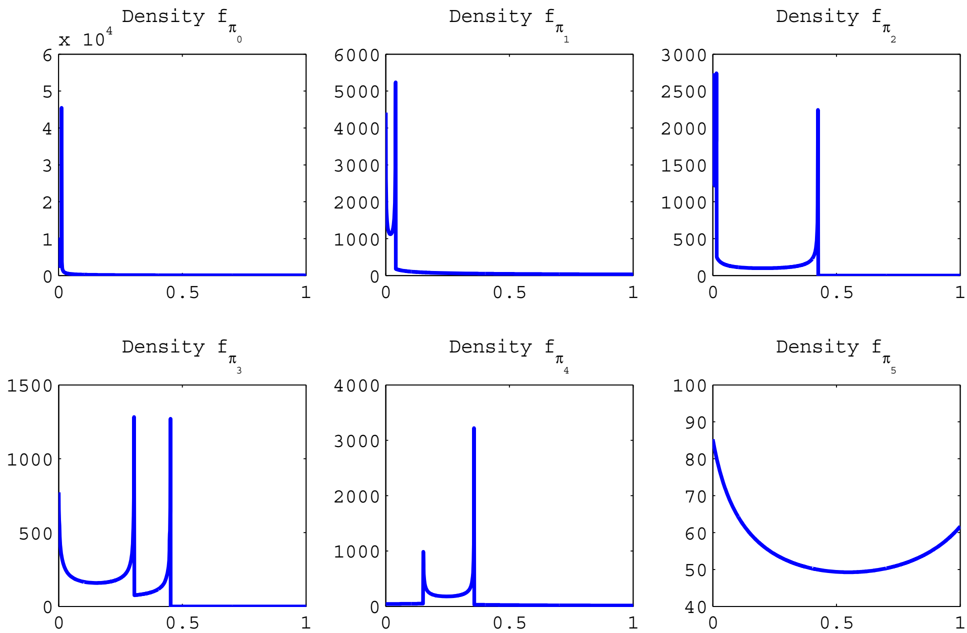


**Figure 5.** Probability density function of stationary distribution of the M/Weibull/1/6 queue with breakdowns and repairs.

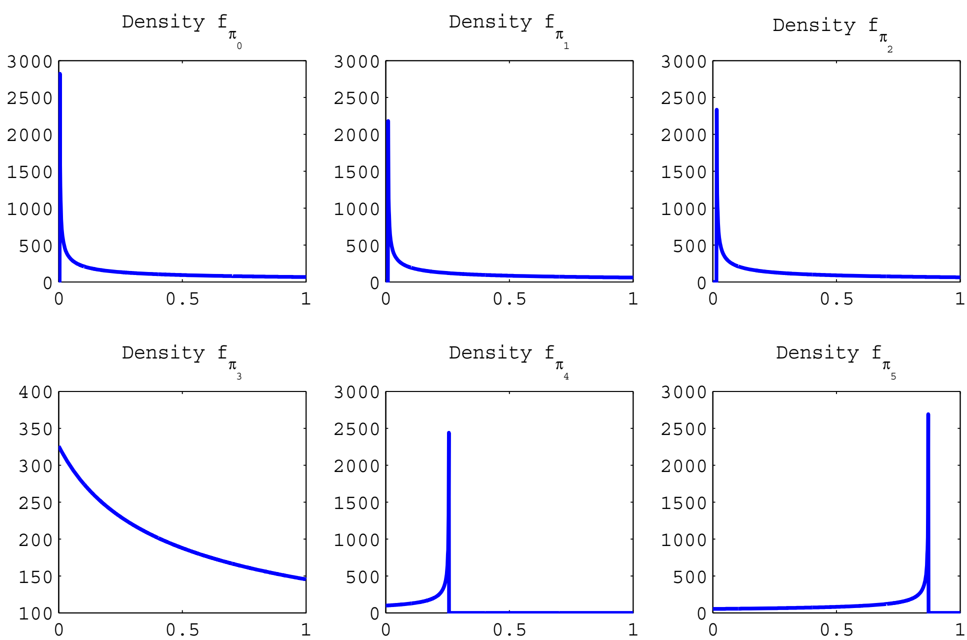


**Figure 6.** Probability density function of stationary distribution of the M/$H_2$/1/6 queue with breakdowns and repairs.

of breakdown is the same for all servers and that the event of a breakdown at a server is independent of everything else. If a breakdown occurs, the customer requesting service is placed back at the front of the queue. The lengths of breakdowns are identically distributed for all servers and follow an exponential distributed with rate $r$. Assume that there can at most be $N$ customers be present in the system (including those in service), and arriving customers that find the queue full are assumed to be blocked and lost to the system. A typical state of this system is denoted by $s = (i, j, k)$, where $i$ is the number of customers waiting for service, $j$ is the number of servers actively serving a customer, and $k$ is the number of servers experiencing a breakdown. Note that $c - j - k$ is the number of severs that is free, and that $i + j$ is the total number of customers in the system.

By assuming that a customer who finds upon arrival more than one server available and that experiences a breakdown during her/his first attempt to receive service, will join the queue and wait until another server becomes available by finishing service a customer or by being repaired before making another attempt to receive service; in other words, a customer that experiences a breakdown does not immediately try again for service but waits for the next event at which a server becomes available for service. With this modification, this queue is seen as a queueing model with a single server. $Q_1$ denotes the infinitesimal generator of the process with breakdown probability equal to 1 and $Q_0$ denotes the infinitesimal generator of the process with no breakdown. Then, $Q_\theta = \theta Q_1 + (1-\theta) Q_0$, for $\theta \in [0, 1]$, yields the infinitesimal generator of the breakdown model, which is given in the following.

For the empty state is holds that

$$Q_\theta(0,0,0;0,1,0) = (1-\theta)\lambda, Q_\theta(0,0,0;1,0,1) = \theta\lambda; \text{ and; } Q_\theta(0,0,0;0,0,0) = -\lambda.$$

For $0 < i$ and $0 < j + k < c$, let

$$\begin{aligned}
Q_\theta(i,j,k;i+1,j,k+1) &= \theta\lambda, \\
Q_\theta(i,j,k;i,j+1,k) &= (1-\theta)\lambda, \\
Q_\theta(i,j,k;i-1,j,k) &= (1-\theta)j\mu, \\
Q_\theta(i,j,k;i,j,k+1) &= \theta j\mu, \\
Q_\theta(i,j,k;i-1,j+1,k-1) &= (1-\theta)kr, \\
Q_\theta(i,j,k;i,j,k) &= \theta kr - (\lambda + j\mu + kr),
\end{aligned}$$

and all zero otherwise.

Let

$$\hat{\lambda} = \lambda + c(\mu + r).$$

Then $Q_\theta$ is uniformizable for all $\theta \in [0, 1]$ with the same rate $\hat{\lambda}$. We now can construct the *sampled chain*, that is, the chain embedded at Poisson $\hat{\lambda}$ time epochs, by $P_\theta = I - (1/\hat{\lambda})Q_\theta$. It is well known that the stationary distribution of $Q_\theta$ and $P_\theta$ coincide. Therefore, we will apply our robust perturbation analysis in the following to $P_\theta$; see[19].

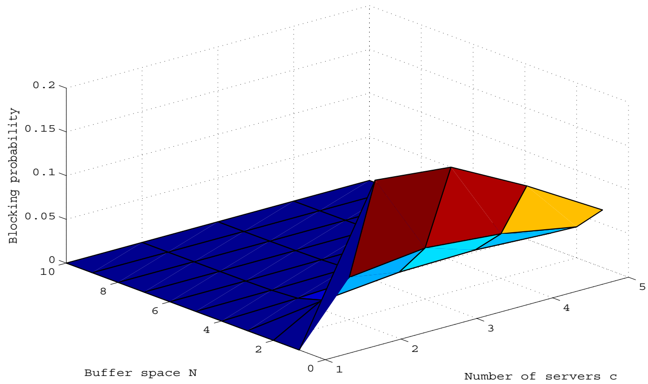


**Figure 7.** Blocking probability versus number of servers $c$ and buffer space $N$.

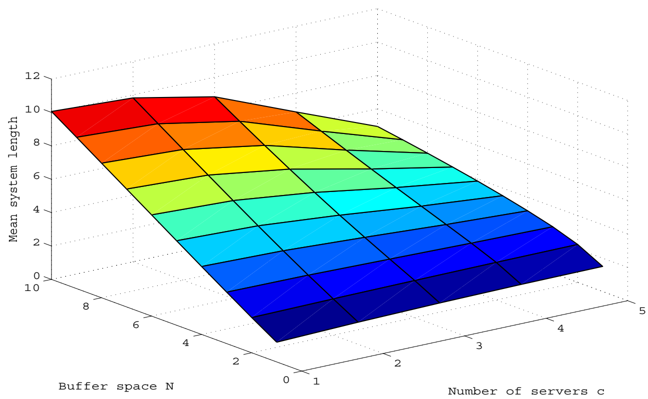


**Figure 8.** Mean system length versus number of servers $c$ and buffer space $N$.

Using the same scheme outlined above, one can easily obtain the quantities of interest of the M/M/c/N queue with breakdowns; especially we are interested in studying the sensitivity of the expected value of blocking probability $\mathbf{E}[\pi_{\theta(\omega)}(N)]$ and that of the average number of customers in the system $\mathbf{E}[L_{\theta(\omega)}]$ with respect to changes in parameter values of number of servers $c$ and buffer space $N$, respectively.

Figure 7 depicts the behavior of the blocking probability with respect to changes of the number of servers $c$ and buffer space $N$. For this, we have set the following parameters: the inter-arrival rate $\lambda = 4$, the service rate $\mu = 2.5$, the repair rate $r = 1.5$, the mean of the breakdown probability $\bar{\theta} = 0.5$, the standard deviation of $\theta$ is $\sigma = 0.09$, and the random variable $\varepsilon(\omega)$ is assumed to follow the standard normal distribution, that is, $\varepsilon(\omega) \sim \mathcal{N}(0, 1)$.

From the obtained numerical results in Figure 7, it is easy to see that increasing the buffer space $N$ leads to decrease the expected value of blocking probability $\mathbf{E}[\pi_{\theta(\omega)}(N)]$. On the other hand, with the increase of the number of servers $c$, the buffer space reduces, then the mean of blocking probability $\mathbf{E}[\pi_{\theta(\omega)}(N)]$ increases.

Figure 8 provides the average number of customers in the system $\mathbf{E}[L_{\theta(\omega)}]$ with a change of the number of servers $c$ and buffer space $N$. All the parameters remain the same as those taken from Figure 7. It is observed that the average number of customers in the system $\mathbf{E}[L_{\theta(\omega)}]$ becomes bigger as the buffer space $N$ increases, but it decreases with the increase of the number of servers $c$.

## 4. Conclusion

In this article, we proposed a framework for propagating epistemic input uncertainty with the help of well-known technique "*power series expansion*" to the output of a model. Specifically, we addressed the problem of computing the uncertainty in performance measures of Markov chain models due to epistemic uncertainty in the model input parameters. The developed approach relates the scaled perturbation analysis to the uncertainty analysis in same framework. For that we proposed an efficient computational algorithm for quantifying the performance measures of considered models by propagating the parameter uncertainty through the power series expansion. The efficiency of the proposed algorithm has been illustrated in two examples: M/G/1/N queue and M/M/c/N queues with breakdowns and repairs. The proposed algorithm is capable of predicting the performance measures accurately in a complete and univocal way by estimating their probability density functions, and directly estimating some quantities of interest. More specifically, we are interested on computing the mean and the variance of the underlying performances. This study also included the estimation of the radius of convergence and the remainder term of the power series. Several numerical experiments are shown to illustrate the performance of the proposed algorithm and are compared with the corresponding Monte Carlo simulations ones. Further studies will include modification of the proposed algorithm by considering directional perturbation analysis of Markov chains. Our methodology is also suitable for studying more complex systems.

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